

Necessary and sufficient conditions for the fractional calculus of variations with Caputo derivatives*

Ricardo Almeida
ricardo.almeida@ua.pt

Delfim F. M. Torres
delfim@ua.pt

Department of Mathematics
University of Aveiro
3810-193 Aveiro, Portugal

Abstract

We prove optimality conditions for different variational functionals containing left and right Caputo fractional derivatives. A sufficient condition of minimization under an appropriate convexity assumption is given. An Euler–Lagrange equation for functionals where the lower and upper bounds of the integral are distinct of the bounds of the Caputo derivative is also proved. Then, the fractional isoperimetric problem is formulated with an integral constraint also containing Caputo derivatives. Normal and abnormal extremals are considered.

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1 Introduction

Fractional Calculus is the branch of Mathematics that generalizes the derivative and the integral of a function to a non-integer order. The subject is not recent and it is as old as the calculus itself. In a letter dated 30th September 1695, Leibniz proposed the following problem to L’Hopital: ”Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?” Since then, several mathematicians studied this question, among them Liouville, Riemann, Weyl and Letnikov. There are many fields of applications where we can use the fractional calculus, like viscoelasticity, electrochemistry, diffusion processes, control theory, heat conduction, electricity, mechanics, chaos

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and fractals (see some references at the end, e.g., [10, 17, 21, 22, 24, 27, 30, 31]). To solve fractional differential equations, there exist several methods: Laplace and Fourier transforms, truncated Taylor series, numerical methods, etc. (see [6] and references therein). Recently, a lot of attention has been put on the fractional calculus of variations (see, e.g., [1, 3, 4, 8, 9, 12–14, 19, 20, 25, 26, 28, 29]). We also mention [7], where necessary and sufficient conditions of optimality for functionals containing fractional integrals and fractional derivatives are presented. For results on fractional optimal control see [2, 15]. In the present paper we work with the Caputo fractional derivative. For problems of calculus of variations with boundary conditions, Caputo's derivative seems to be more natural, since for a given function y to have continuous Riemann–Liouville fractional derivative on a closed interval $[a, b]$, the function must satisfy the conditions $y(a) = y(b) = 0$ [8]. We also mention that, if $y(a) = 0$, then the left Riemann–Liouville derivative is equal to the left Caputo derivative.

The paper is organized as follows. In Section 2 we present the necessary definitions and some necessary facts about fractional calculus. Section 3 is dedicated to our original results. We study fractional Euler–Lagrange equations and the fractional isoperimetric problem within Caputo's fractional derivative context for different kinds of functionals. We also give sufficient conditions of optimality for fractional variational problems.

2 Preliminaries

2.1 Review on fractional calculus

There are several definitions of fractional derivatives and fractional integrals, like Riemann–Liouville, Caputo, Riesz, RieszCaputo, Weyl, Grunwald–Letnikov, Hadamard, Chen, etc. We will present the definitions of the first two of them. Except otherwise stated, proofs of results may be found in [18].

Let $f : [a, b] \rightarrow \mathbb{R}$ be a function, α a positive real number, n the integer satisfying $n - 1 \leq \alpha < n$, and Γ the Euler gamma function. Then,

1. the left and right Riemann–Liouville fractional integrals of order α are defined by

$${}_a I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt,$$

and

$${}_x I_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt,$$

respectively;

2. the left and right Riemann–Liouville fractional derivatives of order α are defined by

$${}_a D_x^\alpha f(x) = \frac{d^n}{dx^n} {}_a I_x^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t) dt,$$

and

$${}_x D_b^\alpha f(x) = (-1)^n \frac{d^n}{dx^n} {}_x I_b^{n-\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (t-x)^{n-\alpha-1} f(t) dt,$$

respectively;

3. the left and right Caputo fractional derivatives of order α are defined by

$${}_a^C D_x^\alpha f(x) = {}_a I_x^{n-\alpha} \frac{d^n}{dx^n} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt,$$

and

$${}_x^C D_b^\alpha f(x) = (-1)^n {}_x I_b^{n-\alpha} \frac{d^n}{dx^n} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b (-1)^n (t-x)^{n-\alpha-1} f^{(n)}(t) dt,$$

respectively.

There exists a relation between the Riemann–Liouville and the Caputo fractional derivatives:

$${}_a^C D_x^\alpha f(x) = {}_a D_x^\alpha f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (x-a)^{k-\alpha}$$

and

$${}_x^C D_b^\alpha f(x) = {}_x D_b^\alpha f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-x)^{k-\alpha}.$$

Therefore,

$$\text{if } f(a) = f'(a) = \dots = f^{(n-1)}(a) = 0, \text{ then } {}_a^C D_x^\alpha f(x) = {}_a D_x^\alpha f(x)$$

and

$$\text{if } f(b) = f'(b) = \dots = f^{(n-1)}(b) = 0, \text{ then } {}_x^C D_b^\alpha f(x) = {}_x D_b^\alpha f(x).$$

These fractional operators are linear, i.e.,

$$\mathcal{P}(\mu f(x) + \nu g(x)) = \mu \mathcal{P}f(x) + \nu \mathcal{P}g(x),$$

where \mathcal{P} is ${}_a D_x^\alpha, {}_x D_b^\alpha, {}_a^C D_x^\alpha, {}_x^C D_b^\alpha, {}_a I_x^\alpha$ or ${}_x I_b^\alpha$, and μ and ν are real numbers.

If $f \in C^n[a, b]$, then the left and right Caputo derivatives are continuous on $[a, b]$. The main advantage of Caputo's approach is that the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer-order differential equations.

Some properties valid for integer differentiation and integer integration remain valid for fractional differentiation and fractional integration; namely the Caputo fractional derivatives and the Riemann–Liouville fractional integrals are inverse operations:

1. If $f \in L_\infty(a, b)$ or $f \in C[a, b]$, and if $\alpha > 0$, then

$${}_a^C D_x^\alpha I_x^\alpha f(x) = f(x) \quad \text{and} \quad {}_x^C D_b^\alpha I_b^\alpha f(x) = f(x).$$

2. If $f \in C^n[a, b]$ and if $\alpha > 0$, then

$${}_a I_x^{\alpha C} D_x^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

and

$${}_x I_b^{\alpha C} D_b^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{k!} (b-x)^k.$$

We also need for our purposes integration by parts formulas. For $\alpha > 0$, we have (cf. [4])

$$\begin{aligned} \int_a^b g(x) \cdot {}_a^C D_x^\alpha f(x) dx &= \int_a^b f(x) \cdot {}_x D_b^\alpha g(x) dx \\ &\quad + \sum_{j=0}^{n-1} \left[{}_x D_b^{\alpha+j-n} g(x) \cdot {}_x D_b^{n-1-j} f(x) \right]_a^b \end{aligned}$$

and

$$\begin{aligned} \int_a^b g(x) \cdot {}_x^C D_b^\alpha f(x) dx &= \int_a^b f(x) \cdot {}_a D_x^\alpha g(x) dx \\ &\quad + \sum_{j=0}^{n-1} \left[(-1)^{n+j} {}_a D_x^{\alpha+j-n} g(x) \cdot {}_a D_x^{n-1-j} f(x) \right]_a^b, \end{aligned}$$

where ${}_a D_x^k g(x) = {}_a I_x^{-k} g(x)$ and ${}_x D_b^k g(x) = {}_x I_b^{-k} g(x)$ if $k < 0$.

Therefore, if $0 < \alpha < 1$, we obtain

$$\int_a^b g(x) \cdot {}_a^C D_x^\alpha f(x) dx = \int_a^b f(x) \cdot {}_x D_b^\alpha g(x) dx + [{}_x I_b^{1-\alpha} g(x) \cdot f(x)]_a^b \quad (1)$$

and

$$\int_a^b g(x) \cdot {}_x^C D_b^\alpha f(x) dx = \int_a^b f(x) \cdot {}_a D_x^\alpha g(x) dx - [{}_a I_x^{1-\alpha} g(x) \cdot f(x)]_a^b. \quad (2)$$

Moreover, if f is a function such that $f(a) = f(b) = 0$, we have simpler formulas:

$$\int_a^b g(x) \cdot {}_a^C D_x^\alpha f(x) dx = \int_a^b f(x) \cdot {}_x D_b^\alpha g(x) dx \quad (3)$$

and

$$\int_a^b g(x) \cdot {}_x^C D_b^\alpha f(x) dx = \int_a^b f(x) \cdot {}_a D_x^\alpha g(x) dx. \quad (4)$$

Remark 1. As α goes to 1, expressions (1) and (2) reduce to the classical integration by parts formulas:

$$\int_a^b g(x) \cdot f'(x) dx = - \int_a^b f(x) \cdot g'(x) dx + [g(x) \cdot f(x)]_a^b$$

and

$$\int_a^b g(x) \cdot (-f'(x)) dx = \int_a^b f(x) \cdot g'(x) dx - [g(x) \cdot f(x)]_a^b,$$

respectively.

Remark 2. Observe that the left member of equations (3) and (4) contains a Caputo fractional derivative, while the other one contains a Riemann–Liouville fractional derivative. However, since there exists a relation between the two derivatives, we could present this formula with only one fractional derivative, although in this case the resulting equation will contain some extra terms.

2.2 Fractional Euler–Lagrange equations

From now on we fix $0 < \alpha, \beta < 1$. Also, to simplify, we denote

$$[y](x) := (x, y(x), {}^C D_x^\alpha y(x), {}^C D_b^\beta y(x)).$$

Let

$${}^\alpha E_b^\beta = \left\{ y : [a, b] \rightarrow \mathbb{R} \mid {}^C D_x^\alpha y \text{ and } {}^C D_b^\beta y \text{ exist and are continuous on } [a, b] \right\}.$$

Definition 2.1. The space of variations ${}^C Var(a, b)$ for the Caputo derivatives is defined by

$${}^C Var(a, b) = \left\{ h \in {}^\alpha E_b^\beta \mid h(a) = h(b) = 0 \right\}.$$

We now present first order necessary conditions of optimality for functionals, defined on ${}^\alpha E_b^\beta$, of the type

$$J(y) = \int_a^b L[y](x) dx. \quad (5)$$

We assume that the map $(x, y, u, v) \rightarrow L(x, y, u, v)$ is a function of class C^1 . Denoting by $\partial_i L$ the partial derivative of L with respect to the i th variable, $i = 1, 2, 3, 4$, we also assume that $\partial_3 L$ has continuous right Riemann–Liouville fractional derivative of order α and $\partial_4 L$ has continuous left Riemann–Liouville fractional derivative of order β .

Definition 2.2. We say that y is a local minimizer (respectively local maximizer) of J if there exists a $\delta > 0$ such that $J(y) \leq J(y_1)$ (respectively $J(y) \geq J(y_1)$) for all y_1 such that $\|y - y_1\| < \delta$.

In [1] Agrawal considers the problem of finding extremals for functionals containing left and right Riemann–Liouville fractional derivatives of the form

$$J(y) = \int_a^b L(x, y(x), {}_aD_x^\alpha y(x), {}_xD_b^\beta y(x)) dx$$

and he derived an Euler–Lagrange equation for an extremum y of J , subject to the boundary conditions $y(a) = y_a$, $y(b) = y_b$:

$$\partial_2 L + {}_xD_b^\alpha \partial_3 L + {}_aD_x^\beta \partial_4 L = 0, \quad \text{for all } x \in [a, b].$$

In [9] a new type of functional is studied, in case where the lower bound of the integral do not coincide with the lower bound of the fractional derivative:

$$J^*(y) = \int_A^B L(x, y(x), {}_aD_x^\alpha y(x)) dx,$$

where $[A, B] \subset [a, b]$. We also mention [3], where an Euler–Lagrange equation and a transversality condition are given, for functionals with left Caputo derivatives and a boundary condition on the initial point $x = a$.

Theorem 2.3 ([3]). *Let J be the functional defined by*

$$J(y) = \int_a^b L(x, y(x), {}^C_aD_x^\alpha y(x)) dx.$$

Let y be a local minimizer of J satisfying the boundary condition $y(a) = y_0$. Then, y satisfies the following conditions:

$$\partial_2 L + {}_xD_b^\alpha \partial_3 L = 0 \tag{6}$$

and

$${}_xI_b^{1-\alpha} \partial_3 L|_{x=b} = 0. \tag{7}$$

Note that the Euler–Lagrange equation (6) contains a Riemann–Liouville fractional derivative, although J has only Caputo’s derivative. Also, the transversality condition (7), in general, contains fractional derivative terms. Thus, in order to solve a fractional variational problem, it may be required fractional boundary conditions. This result is then proven for functionals with higher order fractional derivatives. In [5] functionals with the left and right Caputo fractional derivatives are considered. We include here a short proof. For more on the subject we refer the reader to [11, 16, 23].

Theorem 2.4 ([5]). *Let J be the functional as in (5) and y a local minimizer of J satisfying the boundary conditions $y(a) = y_a$ and $y(b) = y_b$. Then, y satisfies the Euler–Lagrange equation*

$$\partial_2 L + {}_xD_b^\alpha \partial_3 L + {}_aD_x^\beta \partial_4 L = 0. \tag{8}$$

Proof. Let ϵ be a small real parameter and $\eta \in {}^CVar(a, b)$. Consider a variation of y ; say $y + \epsilon\eta$. Since the Caputo derivative operators are linear, we have

$$J(y + \epsilon\eta) = \int_a^b L(x, y + \epsilon\eta, {}^C D_x^\alpha y + \epsilon {}^C D_x^\alpha \eta, {}^C D_b^\beta y + \epsilon {}^C D_b^\beta \eta) dx.$$

We can regard J as a function of one variable, $\hat{J}(\epsilon) = J(y + \epsilon\eta)$. Since y is the local minimizer, \hat{J} attains an extremum at $\epsilon = 0$. Differentiating $\hat{J}(\epsilon)$ at zero, it follows that

$$\int_a^b \left[\partial_2 L \eta + \partial_3 L {}^C D_x^\alpha \eta + \partial_4 L {}^C D_b^\beta \eta \right] dx = 0.$$

Integrating by parts, and since $\eta(a) = \eta(b) = 0$, one finds that

$$\int_a^b \left[\partial_2 L + {}_x D_b^\alpha \partial_3 L + {}_a D_x^\beta \partial_4 L \right] \eta dx = 0$$

for all $\eta \in {}^CVar(a, b)$. By the arbitrariness of η and by the fundamental lemma of the calculus of variations (see, e.g., [32, p. 32]), it follows that

$$\partial_2 L + {}_x D_b^\alpha \partial_3 L + {}_a D_x^\beta \partial_4 L = 0.$$

□

When the term ${}_x D_b^\beta y$ is not present in the function L , then equation (8) reduces to a simpler one:

$$\partial_2 L + {}_x D_b^\alpha \partial_3 L = 0.$$

Moreover, if we allow $\alpha = 1$, and since in that case the right Riemann–Liouville fractional derivative is equal to $-d/dx$, we obtain the classical Euler–Lagrange equation:

$$\partial_2 L - \frac{d}{dx} \partial_3 L = 0.$$

3 Main results

Our first contribution is to generalize Theorem 2.4 (cf. Theorem 3.1 below). We consider a new type of functional, where the lower bound of the integral is greater than the lower bound of the Caputo’s derivative, and the upper bound of the integral is less than the upper bound of the Caputo’s derivative. Because of this, we can not apply directly the integration by parts formula and some technical auxiliary procedures are required. A similar problem is addressed for the Riemann–Liouville fractional derivatives in [9].

3.1 The Euler–Lagrange equation

We consider the functional

$$J^*(y) = \int_A^B L(x, y(x), {}^C D_x^\alpha y(x), {}^C D_b^\beta y(x)) dx, \quad (9)$$

where $(x, y, u, v) \rightarrow L(x, y, u, v) \in C^1$ and $[A, B] \subset [a, b]$.

Theorem 3.1. *If y is a local minimizer of J^* given by (9), satisfying the boundary conditions $y(a) = y_a$ and $y(b) = y_b$, then y satisfies the system*

$$\begin{cases} \partial_2 L + {}_x D_B^\alpha \partial_3 L + {}_A D_x^\beta \partial_4 L = 0 & \text{for all } x \in [A, B] \\ {}_x D_B^\alpha \partial_3 L - {}_x D_A^\alpha \partial_3 L = 0 & \text{for all } x \in [a, A] \\ {}_A D_x^\beta \partial_4 L - {}_B D_x^\beta \partial_4 L = 0 & \text{for all } x \in [B, b] \end{cases}.$$

Proof. Let y be a minimizer and let $\hat{y} = y + \epsilon \eta$ be a variation of y , $\eta \in {}^C Var(a, b)$, such that $\eta(A) = \eta(B) = 0$. Define the new function $\hat{J}^*(\epsilon) = J^*(y + \epsilon \eta)$. By hypothesis, y is a local extremum of J^* and so \hat{J}^* has a local extremum at $\epsilon = 0$. Therefore, the following holds:

$$\begin{aligned} 0 &= \int_A^B [\partial_2 L \eta + \partial_3 L {}^C D_x^\alpha \eta + \partial_4 L {}^C D_b^\beta \eta] dx \\ &= \int_A^B \partial_2 L \eta dx + \left[\int_a^B \partial_3 L {}^C D_x^\alpha \eta dx - \int_a^A \partial_3 L {}^C D_x^\alpha \eta dx \right] \\ &\quad + \left[\int_A^b \partial_4 L {}^C D_b^\beta \eta dx - \int_B^b \partial_4 L {}^C D_b^\beta \eta dx \right]. \end{aligned}$$

Integrating by parts the four last terms gives:

$$\begin{aligned} 0 &= \int_A^B \partial_2 L \eta dx + \left[\int_a^B \eta {}_x D_B^\alpha \partial_3 L dx - \int_a^A \eta {}_x D_A^\alpha \partial_3 L dx \right] \\ &\quad + \left[\int_A^b \eta {}_A D_x^\beta \partial_4 L dx - \int_B^b \eta {}_B D_x^\beta \partial_4 L dx \right] \\ &= \int_A^B \partial_2 L \eta dx + \left[\int_a^A \eta {}_x D_B^\alpha \partial_3 L dx + \int_A^B \eta {}_x D_B^\alpha \partial_3 L dx - \int_a^A \eta {}_x D_A^\alpha \partial_3 L dx \right] \\ &\quad + \left[\int_A^B \eta {}_A D_x^\beta \partial_4 L dx + \int_B^b \eta {}_A D_x^\beta \partial_4 L dx - \int_B^b \eta {}_B D_x^\beta \partial_4 L dx \right] \\ &= \int_a^A [{}_x D_B^\alpha \partial_3 L - {}_x D_A^\alpha \partial_3 L] \eta dx + \int_A^B [\partial_2 L + {}_x D_B^\alpha \partial_3 L + {}_A D_x^\beta \partial_4 L] \eta dx \\ &\quad + \int_B^b [{}_A D_x^\beta \partial_4 L - {}_B D_x^\beta \partial_4 L] \eta dx. \end{aligned}$$

Since η is an arbitrary function, we can assume that $\eta(x) = 0$ for all $x \in [A, b]$ and so by the fundamental lemma of the calculus of variations,

$${}_x D_B^\alpha \partial_3 L - {}_x D_A^\alpha \partial_3 L = 0, \text{ for all } x \in [a, A].$$

Similarly, one proves the other two conditions:

$$\partial_2 L + {}_x D_B^\alpha \partial_3 L + {}_A D_x^\beta \partial_4 L = 0, \text{ for all } x \in [A, B],$$

and

$${}_A D_x^\beta \partial_4 L - {}_B D_x^\beta \partial_4 L = 0, \text{ for all } x \in [B, b].$$

□

3.2 The fractional isoperimetric problem

The *isoperimetric problem* is one of the most ancient problems of the calculus of variations. For example, given a positive real number l , what is the shape of the closed curve C of length l which defines the maximal area? The most essential contribution towards its rigorous proof was given in 1841 and is due to Jacob Steiner (1796–1863). We state the fractional isoperimetric problem as follows.

Given a functional J as in (5), which functions y minimize (or maximize) J , when subject to given boundary conditions

$$y(a) = y_a, \quad y(b) = y_b, \tag{10}$$

and an integral constraint

$$I(y) = \int_a^b g[y] dx = l. \tag{11}$$

Here, similarly as before, we consider a function g of class C^1 , such that $\partial_3 g$ has continuous right Riemann–Liouville fractional derivative of order α and $\partial_4 g$ has continuous left Riemann–Liouville fractional derivative of order β , and functions $y \in {}_a^\alpha E_b^\beta$. A function $y \in {}_a^\alpha E_b^\beta$ that satisfies (10) and (11) is called admissible.

Definition 3.2. An admissible function y is an *extremal* for I in (11) if it satisfies the equation

$$\partial_2 g[y](x) + {}_x D_b^\alpha \partial_3 g[y](x) + {}_a D_x^\beta \partial_4 g[y](x) = 0, \text{ for all } x \in [a, b].$$

Observe that Definition 3.2 makes sense by Theorem 2.4. To solve the isoperimetric problem, the idea is to consider a new extended function. The exact formula for such extended function depends on y being or not an extremal for the integral functional $I(y)$ (cf. Theorems 3.3 and 3.4).

Theorem 3.3. *Let y be a local minimum for J given by (5), subject to the conditions (10) and (11). If y is not an extremal for the functional I , then there exists a constant λ such that y satisfies*

$$\partial_2 F + {}_x D_b^\alpha \partial_3 F + {}_a D_x^\beta \partial_4 F = 0 \tag{12}$$

for all $x \in [a, b]$, where $F = L + \lambda g$.

Proof. Let $\eta_1, \eta_2 \in {}^CVar(a, b)$ be two functions, ϵ_1 and ϵ_2 two reals, and consider the new function of two parameters

$$\hat{y} = y + \epsilon_1 \eta_1 + \epsilon_2 \eta_2. \quad (13)$$

The reason why we consider two parameters is because we can choose one of them as a function of the other in order to \hat{y} satisfy the integral constraint. Let

$$\hat{I}(\epsilon_1, \epsilon_2) = \int_a^b g[y + \epsilon_1 \eta_1 + \epsilon_2 \eta_2](x) dx - l.$$

It follows by integration by parts that

$$\begin{aligned} \left. \frac{\partial \hat{I}}{\partial \epsilon_2} \right|_{(0,0)} &= \int_a^b \left(\partial_2 g \eta_2 + \partial_3 g {}^C D_x^\alpha \eta_2 + \partial_4 g {}^C D_x^\beta \eta_2 \right) dx \\ &= \int_a^b \left(\partial_2 g + {}_x D_b^\alpha \partial_3 g + {}_a D_x^\beta \partial_4 g \right) \eta_2 dx. \end{aligned}$$

We have assumed that y is not an extremal for I , and therefore there exists a function η_2 satisfying the condition

$$\left. \frac{\partial \hat{I}}{\partial \epsilon_2} \right|_{(0,0)} \neq 0. \quad (14)$$

Using (14) and the fact that $\hat{I}(0, 0) = 0$, by the Implicit Function Theorem there exists a C^1 function $\epsilon_2(\cdot)$, defined in some neighborhood of zero, such that

$$\hat{I}(\epsilon_1, \epsilon_2(\epsilon_1)) = 0.$$

Therefore, there exists a family of variations of type (13) which satisfy the integral constraint.

We will now prove condition (12). Similarly as before, we define a new function of two variables $\hat{J}(\epsilon_1, \epsilon_2) = J(\hat{y})$. Since $(0, 0)$ is a local minimum of \hat{J} , subject to the constraint $\hat{I}(0, 0) = 0$, and $\nabla \hat{I}(0, 0) \neq \mathbf{0}$, by the Lagrange Multiplier Rule (see, e.g., [32, p. 77]), there exists a constant λ for which the following holds:

$$\nabla(\hat{J}(0, 0) + \lambda \hat{I}(0, 0)) = \mathbf{0}.$$

Simple calculations show that

$$\left. \frac{\partial \hat{J}}{\partial \epsilon_1} \right|_{(0,0)} = \int_a^b \left[\partial_2 L + {}_x D_b^\alpha \partial_3 L + {}_a D_x^\beta \partial_4 L \right] \eta_1 dx$$

and

$$\left. \frac{\partial \hat{I}}{\partial \epsilon_1} \right|_{(0,0)} = \int_a^b \left[\partial_2 g + {}_x D_b^\alpha \partial_3 g + {}_a D_x^\beta \partial_4 g \right] \eta_1 dx.$$

In conclusion, it follows that

$$\int_a^b [\partial_2 L + {}_x D_b^\alpha \partial_3 L + {}_a D_x^\beta \partial_4 L + \lambda (\partial_2 g + {}_x D_b^\alpha \partial_3 g + {}_a D_x^\beta \partial_4 g)] \eta_1 dx = 0.$$

By the arbitrariness of η_1 and the fundamental lemma of calculus of variations, one must have

$$\partial_2 L + {}_x D_b^\alpha \partial_3 L + {}_a D_x^\beta \partial_4 L + \lambda (\partial_2 g + {}_x D_b^\alpha \partial_3 g + {}_a D_x^\beta \partial_4 g) = 0.$$

This is equivalent to

$$\partial_2 F + {}_x D_b^\alpha \partial_3 F + {}_a D_x^\beta \partial_4 F = 0.$$

□

Example 1. Let $\bar{y}(x) = E_\alpha(x^\alpha)$, $x \in [0, 1]$, where E_α is the Mittag–Leffler function:

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad x \in \mathbb{R}, \alpha > 0.$$

When $\alpha = 1$, the Mittag–Leffler function is the exponential function, $E_1(x) = e^x$.

The left Caputo fractional derivative of \bar{y} is \bar{y} (cf. [18, p. 98]),

$${}_0^C D_x^\alpha \bar{y} = \bar{y}.$$

Consider the following fractional variational problem:

$$\begin{cases} J(y) = \int_0^1 ({}_0^C D_x^\alpha y)^2 dx \rightarrow \text{extr}, \\ I(y) = \int_0^1 \bar{y} {}_0^C D_x^\alpha y dx = \int_0^1 (\bar{y})^2 dx, \\ y(0) = 1 \quad \text{and} \quad y(1) = E_\alpha(1). \end{cases} \quad (15)$$

The augmented function is

$$F(x, y, {}_0^C D_x^\alpha y, {}_x D_1^\beta y, \lambda) = ({}_0^C D_x^\alpha y)^2 + \lambda \bar{y} {}_0^C D_x^\alpha y$$

and the fractional Euler–Lagrange equation is

$$\partial_2 F + {}_x D_1^\alpha \partial_3 F + {}_x D_1^\beta \partial_4 F = 0$$

i.e.,

$${}_x D_1^\alpha (2 {}_0^C D_x^\alpha y + \lambda \bar{y}) = 0.$$

A solution of this problem is $\lambda = -2$ and $y = \bar{y}$.

Observe that, as $\alpha \rightarrow 1$, the variational problem (15) becomes

$$\int_0^1 y'^2 dx \rightarrow \text{extr},$$

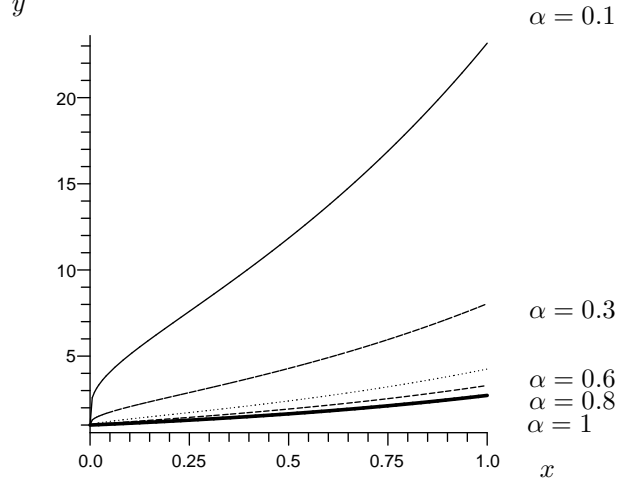


Figure 1: Solutions of problem (15).

$$\int_0^1 \bar{y} y' dx = \frac{1}{2}(e^2 - 1),$$

$$y(0) = 1 \quad \text{and} \quad y(1) = e,$$

and the Euler–Lagrange equation is

$$\partial_2 F - \frac{d}{dx} \partial_3 F = 0 \Leftrightarrow -\frac{d}{dx} (2y' - 2\bar{y}) = 0, \quad (16)$$

where $F = y'^2 - 2\bar{y}y'$. Also, for $\alpha = 1$, $\bar{y}(x) = e^x$, which is obviously a solution of the differential equation (16) (cf. Figure 1).

We now study the case when y is an extremal of I (the so called abnormal case).

Theorem 3.4. *Let y be a local minimum of J (5), subject to the conditions (10) and (11). Then, there exist two constants λ_0 and λ , with $(\lambda_0, \lambda) \neq (0, 0)$, such that*

$$\partial_2 K + {}_x D_b^\alpha \partial_3 K + {}_a D_x^\beta \partial_4 K = 0$$

where $K = \lambda_0 L + \lambda g$.

Proof. Following the proof of Theorem 3.3, $(0, 0)$ is an extremal of \hat{J} subject to the constraint $\hat{I} = 0$. Then, by the abnormal Lagrange multiplier rule (see, e.g., [32, p. 82]), there exist two reals λ_0 and λ , not both zero, such that

$$\nabla(\lambda_0 \hat{J}(0, 0) + \lambda \hat{I}(0, 0)) = \mathbf{0}.$$

Therefore,

$$\lambda_0 \left. \frac{\partial \hat{J}}{\partial \epsilon_1} \right|_{(0,0)} + \lambda \left. \frac{\partial \hat{I}}{\partial \epsilon_1} \right|_{(0,0)} = 0.$$

The rest of proof is similar to the one of Theorem 3.3. \square

3.3 An extension

We now present a solution for the isoperimetric problem for functionals of type (9). Similarly, one has an integral constraint, but this time of the form

$$I^*(y) = \int_A^B g(x, y(x), {}^C D_x^\alpha y(x), {}^C D_b^\beta y(x)) dx = l. \quad (17)$$

Again, we need the concept of extremal for a functional of type (17).

Definition 3.5. A function y is called extremal for I^* given by (17) if

$$\partial_2 g[y](x) + {}_x D_B^\alpha \partial_3 g[y](x) + {}_A D_x^\beta \partial_4 g[y](x) = 0, \text{ for all } x \in [A, B].$$

Theorem 3.6. If y is a local minimum of J^* given by (9), when restricted to the conditions $y(a) = y_a$, $y(b) = y_b$ and (17), and if y is not an extremal for I^* , then there exists a constant λ such that

$$\begin{cases} \partial_2 F + {}_x D_B^\alpha \partial_3 F + {}_A D_x^\beta \partial_4 F = 0 & \text{for all } x \in [A, B] \\ {}_x D_B^\alpha \partial_3 F - {}_x D_A^\alpha \partial_3 F = 0 & \text{for all } x \in [a, A] \\ {}_A D_x^\beta \partial_4 F - {}_B D_x^\beta \partial_4 F = 0 & \text{for all } x \in [B, b] \end{cases} \quad (18)$$

where $F = L + \lambda g$.

Remark 3. In case $[A, B] = [a, b]$, Theorem 3.6 is reduced to Theorem 3.3.

Proof. We consider a variation of form $\hat{y} = y + \epsilon_1 \eta_1 + \epsilon_2 \eta_2$, where

$$\eta_1, \eta_2 \in {}^C Var(a, b) \text{ and } \eta_1(A) = \eta_1(B) = \eta_2(A) = \eta_2(B) = 0.$$

Define \hat{I}^* by the expression

$$\hat{I}^*(\epsilon_1, \epsilon_2) = \int_A^B g[\hat{y}] dx - l.$$

Then, $\hat{I}^*(0, 0) = 0$ and

$$\begin{aligned} \left. \frac{\partial \hat{I}^*}{\partial \epsilon_2} \right|_{(0,0)} &= \int_A^B [\partial_2 g \eta_2 + \partial_3 g {}^C D_x^\alpha \eta_2 + \partial_4 g {}^C D_b^\beta \eta_2] dx \\ &= \int_a^A [{}_x D_B^\alpha \partial_3 g - {}_x D_A^\alpha \partial_3 g] \eta_2 dx + \int_A^B [\partial_2 g + {}_x D_B^\alpha \partial_3 g + {}_A D_x^\beta \partial_4 g] \eta_2 dx \\ &\quad + \int_B^b [{}_A D_x^\beta \partial_4 g - {}_B D_x^\beta \partial_4 g] \eta_2 dx \end{aligned}$$

(the last expression follows by integration by parts and some technical calculations as presented in the proof of Theorem 3.1). Let η_2 be a function such that

$$\left. \frac{\partial \hat{I}^*}{\partial \epsilon_2} \right|_{(0,0)} \neq 0$$

(its existence is guarantied since y is not an extremal for I^*). Therefore, we can consider a subset of the family of functions $\{y + \epsilon_1 \eta_1 + \epsilon_2 \eta_2 \mid (\epsilon_1, \epsilon_2) \in \mathbb{R}^2\}$ that is admissible for the isoperimetric problem. Let $\hat{J}^*(\epsilon_1, \epsilon_2) = J^*(\hat{y})$. Then, there exists a real λ such that

$$\nabla(\hat{J}^*(0, 0) + \lambda \hat{I}^*(0, 0)) = \mathbf{0}. \quad (19)$$

Similarly, one has

$$\begin{aligned} \left. \frac{\partial \hat{J}^*}{\partial \epsilon_1} \right|_{(0,0)} &= \int_a^A [{}_x D_B^\alpha \partial_3 L - {}_x D_A^\alpha \partial_3 L] \eta_1 dx + \int_A^B [\partial_2 L + {}_x D_B^\alpha \partial_3 L + {}_A D_x^\beta \partial_4 L] \eta_1 dx \\ &\quad + \int_B^b [{}_A D_x^\beta \partial_4 L - {}_B D_x^\beta \partial_4 L] \eta_1 dx, \end{aligned}$$

and

$$\begin{aligned} \left. \frac{\partial \hat{I}^*}{\partial \epsilon_1} \right|_{(0,0)} &= \int_a^A [{}_x D_B^\alpha \partial_3 g - {}_x D_A^\alpha \partial_3 g] \eta_1 dx + \int_A^B [\partial_2 g + {}_x D_B^\alpha \partial_3 g + {}_A D_x^\beta \partial_4 g] \eta_1 dx \\ &\quad + \int_B^b [{}_A D_x^\beta \partial_4 g - {}_B D_x^\beta \partial_4 g] \eta_1 dx. \end{aligned}$$

By (19),

$$\left. \frac{\partial \hat{J}^*}{\partial \epsilon_1} \right|_{(0,0)} + \lambda \left. \frac{\partial \hat{I}^*}{\partial \epsilon_1} \right|_{(0,0)} = 0$$

and (18) follows from the arbitrariness of η_1 . \square

The following result generalizes Theorem 3.4 and is proved in a similar way.

Theorem 3.7. *If y is a local minimum of J^* given by (9), subject to the boundary conditions (10) and the integral constraint (17), then there exist two constants λ_0 and λ , not both zero, such that*

$$\begin{cases} \partial_2 K + {}_x D_B^\alpha \partial_3 K + {}_A D_x^\beta \partial_4 K = 0 & \text{for all } x \in [A, B] \\ {}_x D_B^\alpha \partial_3 K - {}_x D_A^\alpha \partial_3 K = 0 & \text{for all } x \in [a, A] \\ {}_A D_x^\beta \partial_4 K - {}_B D_x^\beta \partial_4 K = 0 & \text{for all } x \in [B, b] \end{cases}$$

where $K = \lambda_0 L + \lambda g$.

3.4 Sufficient conditions of optimality

We are now interested in finding sufficient conditions for J to attain local extremes. Typically, some conditions of convexity over the Lagrangian are needed.

Definition 3.8. We say that $f(\underline{x}, y, u, v)$ is convex in $S \subseteq \mathbb{R}^4$ if $\partial_2 f$, $\partial_3 f$ and $\partial_4 f$ exist and are continuous, and the condition

$$\begin{aligned} f(x, y + y_1, u + u_1, v + v_1) - f(x, y, u, v) \\ \geq \partial_2 f(x, y, u, v)y_1 + \partial_3 f(x, y, u, v)u_1 + \partial_4 f(x, y, u, v)v_1 \end{aligned}$$

holds for every $(x, y, u, v), (x, y + y_1, u + u_1, v + v_1) \in S$.

Theorem 3.9. Suppose that the function $L(\underline{x}, y, u, v)$ is convex in $[a, b] \times \mathbb{R}^3$. Then each solution y_0 of the fractional Euler–Lagrange equation (8) minimizes

$$J(y) = \int_a^b L[y]dx,$$

when restricted to the boundary conditions $y(a) = y_0(a)$ and $y(b) = y_0(b)$.

Proof. Let $\eta \in {}_a^\alpha E_b^\beta$ be a function such that $\eta(a) = \eta(b) = 0$. Then, using integration by parts, it follows that

$$\begin{aligned} J(y_0 + \eta) - J(y_0) &= \int_a^b (L[y_0 + \eta] - L[y_0]) dx \\ &\geq \int_a^b \left(\partial_2 L[y_0]\eta + \partial_3 L[y_0]_a^C D_x^\alpha \eta + \partial_4 L[y_0]_x^C D_b^\beta \eta \right) dx \\ &= \int_a^b [\partial_2 L + {}_x D_b^\alpha \partial_3 L + {}_a D_x^\beta \partial_4 L] [y_0] \cdot \eta dx = 0 \end{aligned}$$

since y_0 is a solution of (8). We just proved that $J(y_0 + \eta) \geq J(y_0)$. \square

This procedure can be easily adapted for the isoperimetric problem.

Theorem 3.10. Suppose that, for some constant λ , the functions $L(\underline{x}, y, u, v)$ and $\lambda g(\underline{x}, y, u, v)$ are convex in $[a, b] \times \mathbb{R}^3$. Let $F = L + \lambda g$. Then each solution y_0 of the fractional equation

$$\partial_2 F + {}_x D_b^\alpha \partial_3 F + {}_a D_x^\beta \partial_4 F = 0$$

minimizes

$$J(y) = \int_a^b L[y]dx,$$

under the constraints $y(a) = y_0(a)$ and $y(b) = y_0(b)$ and

$$I(y) = \int_a^b g[y]dx = l, \quad l \in \mathbb{R}.$$

Proof. Let us prove that y_0 minimizes

$$\tilde{F}(y) = \int_a^b (L[y] + \lambda g[y]) dx.$$

First, it is easy to prove that $L(\underline{x}, y, u, v) + \lambda g(\underline{x}, y, u, v)$ is convex. Let $\eta \in {}^\alpha_a E_b^\beta$ be such that $\eta(a) = \eta(b) = 0$. Then, by Theorem 3.9, $\tilde{F}(y_0 + \eta) \geq \tilde{F}(y_0)$. In other words, if $y \in {}^\alpha_a E_b^\beta$ is any function such that $y(a) = y_0(a)$ and $y(b) = y_0(b)$, then

$$\int_a^b L[y] dx + \int_a^b \lambda g[y] dx \geq \int_a^b L[y_0] dx + \int_a^b \lambda g[y_0] dx.$$

If we restrict to the integral constraint, we obtain

$$\int_a^b L[y] dx + l \geq \int_a^b L[y_0] dx + l,$$

and so

$$\int_a^b L[y] dx \geq \int_a^b L[y_0] dx,$$

proving the desired result. \square

Example 2. Recall Example 1. Since $L(\underline{x}, y, u, v) = u^2$ and $\lambda g(\underline{x}, y, u, v) = -2\bar{\gamma}u$ are both convex, we conclude that $\bar{\gamma}$ is actually a minimum for the fractional variational problem (15).

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